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Navigation with High-Altitude Satellites: A Study of the Effects of Satellite-User Geometry on Position Accuracy

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY

LINCOLN LABORATORY

NAVIGATION WITH HIGH-ALTITUDE SATELLITES: A STUDY OF THE EFFECTS OF SATELLITE-USER GEOMETRY ON POSITION ACCURACY

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ABSTRACT

An error analysis of an hyperbolic navigation system using high-altitude satellites revealed that certain satellite arrangements result in long, narrow corridors on the earth (singular regions), in which navigation errors are very large. The study showed that in these singular regions the satellite-user geometry results in navigation equations which are sensitive to measurement errors and thus cannot be solved accurately for all three user coordinates.

Accepted for the Air Force
Franklin C. Hudson
Chief, Lincoln Laboratory Office

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I. INTRODUCTION

It has become apparent that high-altitude satellites have possibilities for navigation as well as communication. Although use of such satellites can allow the user to obtain an accurate position fix, Schweppe* pointed out that, for certain arrangements of the satellites and the user, the error in the user's determination of his position in three coordinates is extremely large. It is the purpose of this report to determine the extent of such singular regions, i.e., regions in which a user cannot make a unique three-coordinate fix, and to explore the reasons for their existence.

II. SYSTEM CONCEPT

The navigation system considered is an hyperbolic system in which a user receives timing signals transmitted from each of three synchronous-altitude satellites. The satellite clocks, from which the timing signals are derived, are assumed to be perfectly synchronized to a master clock. The user measures his height as well as the time of arrival of each of the satellite signals. In the noiseless case (that is, the user has made all his measurements perfectly and knows the satellite positions exactly), by using the difference in the time of arrival of the timing signals emitted by two of these satellites, the user can position himself on the locus of points satisfying this condition, that is, on an hyperbola of revolution. By taking one of these satellites and a third, he can determine another hyperboloid. The intersection of these two figures is not sufficient to determine the user's position in three coordinates. The user knows, however, he is on the surface of a sphere with radius equal to the radius of the earth plus his measured height. The intersection of the sphere and the hyperboloids will determine his position in three coordinates.

In practice, the user is not able to make perfect measurements and, therefore, the effects of measurement errors must be considered.

III. ERROR ANALYSIS

The quantities observed by the user are (a) the time of arrival of timing signals from each of the three satellites (as indicated on the user's clock) and (b) the user's height or distance from the center of the earth. Denote the user's position in three-dimensional space (origin at the center of the earth) by $\underline{p}_u$, a three-dimensional column vector (Fig. 1). Similarly, denote the satellite positions by the vectors $\underline{p}_1$, $\underline{p}_2$, $\underline{p}_3$. If the user's clock were synchronized to the master

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* F. C. Schweppe (private communication).

clock governing the satellite transmissions and there were no errors in the user's measurements, the time of arrival of a signal from a satellite would provide the user with the distance between himself and the satellite. That is, the user would observe indirectly

$$s_j = |\underline{p}_u - \underline{p}_j| \quad , \quad j = 1, 2, 3 \quad .$$

If the user clock were offset from the master clock by an unknown constant τ_o, the user would really observe

$$|\underline{p}_u - \underline{p}_j| + \tau_o = s_j + \tau_o \quad , \quad j = 1, 2, 3$$

where τ_o has the dimension of length, since it has the effect of changing the apparent distance between the user and the satellite. The user height is

$$h_u = |\underline{p}_u|$$

and in the absence of any noise, the observables are

$$\underline{y} = \begin{bmatrix} s_1 + \tau_o \\ s_2 + \tau_o \\ s_3 + \tau_o \\ h_u \end{bmatrix} \quad . \tag{1}$$

The quantities that $\underline{y}$ depends on then are $\underline{p}_u$, $\underline{p}_1$, $\underline{p}_2$, $\underline{p}_3$, and τ_o. We express these quantities as a column vector, called the state vector $\underline{x}$,

$$\underline{x} = \begin{bmatrix} \underline{p}_u \\ \underline{p}_1 \\ \underline{p}_2 \\ \underline{p}_3 \\ \tau_o \end{bmatrix} \tag{2}$$

so that, as long as there is no noise,

$$\underline{y} = \underline{h}(\underline{x}) = \begin{bmatrix} h_1(\underline{x}) \\ h_2(\underline{x}) \\ h_3(\underline{x}) \\ h_4(\underline{x}) \end{bmatrix} = \begin{bmatrix} |\underline{p}_u - \underline{p}_1| + \tau_o \\ |\underline{p}_u - \underline{p}_2| + \tau_o \\ |\underline{p}_u - \underline{p}_3| + \tau_o \\ |\underline{p}_u| \end{bmatrix} \quad . \tag{3}$$

The observables are, however, obscured by additive noise which can be expressed in vector form as

$$\underline{r} = \underline{y} + \underline{n} = \underline{h}(\underline{x}) + \underline{n} \quad . \tag{4}$$

We assume that the a priori knowledge of the components of the state vector is in the form of a probability density. In particular, we take $\underline{x}$ to have statistically independent Gaussian components with mean

$$E[\underline{x}] = \underline{x}_m$$

and covariance

$$W = E[(\underline{x} - \underline{x}_m)(\underline{x} - \underline{x}_m)']$$

where ( )' denotes matrix transpose and E denotes the statistical expectation.* The noise vector $\underline{n}$ is also taken to be Gaussian with zero mean. Under these assumptions, the problem of calculating the user position from noisy observables can be viewed as forming a statistical estimate of the state vector $\underline{x}$. In particular, the maximum a posteriori probability estimate of the state vector is formed, i.e., the estimate of $\underline{x}$ that maximizes $p(\underline{x}|\underline{r})$. Let $\hat{\underline{x}}$ denote this estimate. The error analysis of such a navigation scheme consists of calculating the covariance matrix of the error vector $\underline{e} = (\underline{x} - \hat{\underline{x}})$.

In order to do this, the vector function $\underline{h}(\underline{x})$ is first linearized about the user's true position to make this a linear estimation problem. Let the matrix H denote the linear transformation required to obtain the linearized version of the general nonlinear function $\underline{h}(\cdot)$, i.e.,

$$\underline{h}(\underline{x}) \approx \underline{h}(\underline{x}_o) + H(\underline{x} - \underline{x}_o) \tag{5}$$

where

$$H = (h_{ij}) = \left(\frac{\partial h_i(\underline{x})}{\partial x_j}\right)_{(\underline{x}=\underline{x}_o)} .$$

This linear approximation of the nonlinear function $\underline{h}(\cdot)$ is accurate when the difference between the estimate $\hat{\underline{x}}$ and $\underline{x}_o$ is small, as it should be for an accurate navigation system.

The resulting error covariance matrix

$$V = (v_{ij}) = E[\underline{e}\underline{e}']$$

has been calculated as†

$$V = W - WH[N + HWH']^{-1} HW = [H'N^{-1} H + W^{-1}]^{-1} \tag{6}$$

where W is as previously defined and N is the covariance matrix of the noise vector, i.e.,

$$N = (n_{ij}) = E[n_i n_j]$$

where n_k is an element of the noise vector $\underline{n}$.

Since we are interested in the estimate of $\underline{p}_u$, only a submatrix of V is pertinent to the actual navigation errors. The upper left 3 × 3 submatrix of V is the error covariance matrix of the components of the position estimate $\hat{\underline{p}}_u$. As our measure of the accuracy of navigation, we adopt the root-mean-squared (RMS) error between the estimate $\hat{\underline{p}}_u$ and the true position $\underline{p}_u$.

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* The statistical expectation of a matrix is taken to be the matrix of expected values of each of the elements.

†D. L. Snyder, "Navigation with High-Altitude Satellites: A Study of the Errors in Position Determination," Technical Note 1967-11, Lincoln Laboratory, M.I.T. (6 February 1967), DDC 648828, H-803.

A computer program was written to calculate the V matrix and from it the RMS position error for any positions of the satellites and the user. The input to the program consists of the *a priori* standard deviation of the components of the state vector $\underline{x}$ and the observation noise $\underline{n}$. These are taken to be

Ranging error	10 meters (RMS)
Height error	10 meters (RMS)
User position error	10^6 meters (RMS)
Satellite position error	10^2 meters (RMS)
Clock error	10^6 meters (RMS)

where the square of the ranging error and the square of the height error are the diagonal elements of the covariance matrix N, the other elements of this matrix being zero, since the observation errors are taken to be statistically independent. The squares of the remaining three parameters are the diagonal elements of the W matrix, which is also diagonal for the same reason. In order to eliminate any *a priori* knowledge of user position and clock error, the variances of these quantities should approach infinity. This limit was well approximated by using 10^6 meters as the *a priori* variance of these quantities, which proved to be of inestimable value in view of the finite word length of the digital computer used for the calculations. The other parameters lead to reasonable navigation accuracies and are realizable (hopefully) with practical equipment.* The program calculates the RMS error between the maximum *a posteriori* estimate of $\underline{p}_u$ and the actual user's position for any user-satellite geometries.

Consider the case where the satellite positions are as follows:

	Latitude	Longitude
Satellite 1	0.0	30.0
Satellite 2	0.0	0.0
Satellite 3	0.0	−30.0

that is, when all three satellites are along the equator. The RMS errors for the user at various positions between 60° latitude and the equator, and 40° longitude and 0° longitude are given in Fig. 2. At 60° latitude, the RMS error is of the order of 10^2 meters for all longitudes and increases steadily as the user approaches the equator; at the equator, it has a value of 10^6 meters, but this is the *a priori* assumption of the position error (Fig. 3). The observations and calculations have not given the user any information about his position. This is a singular region of the type discovered by Schweppe.

We have seen that the RMS error varies markedly over the different user positions. The question then arises as to why certain arrangements of satellites and user cause large errors in position determination.

IV. ANGLE BETWEEN HYPERBOLAS

Consider again how a user would determine his position if this were a deterministic case (i.e., no noise and no errors in satellite position knowledge). He would calculate two hyperbolas of revolution and a sphere, all three of which he knows he must be on (Fig. 4). The intersection of these figures, therefore, provides his position fix. Suppose, however, instead of being a

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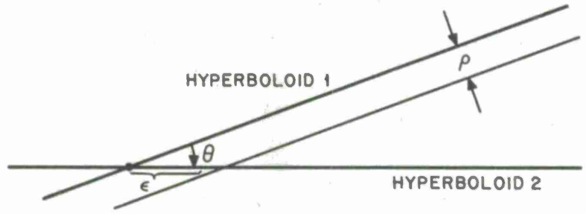
* T. J. Goblick, Jr., "Navigation with High-Altitude Satellites: A Study of Ranging Errors," Technical Note 1966-46, Lincoln Laboratory, M.I.T. (26 August 1966), DDC 643851, H-752.

deterministic case, there is some error in the user's measurements and thus an error ρ in his calculations of the position of one hyperbola. This error would produce an error in position determination

$$\epsilon = \frac{\rho}{\sin\Theta} \tag{7}$$



where Θ denotes the angle formed by the hyperboloids intersected with the plane of the earth at the point of the user. If Θ is close to 90°, $\sin\Theta$ is close to one and the error ϵ in the user's determination of his position is nearly equal to the error ρ in his calculations. However, as $\Theta \rightarrow 0$, $\sin\Theta \rightarrow 0$ and $\epsilon \rightarrow \infty$, which indicates that the error ρ would be greatly magnified for small angles. The hyperboloids and the sphere intersect in such a way that the user, unable to measure perfectly, cannot determine his position accurately in three coordinates. The error in his position determination would be entirely dependent on his a priori knowledge.

To calculate Θ, let $\underline{p}_u$ be a three-dimensional vector (as in Sec. III), origin at the center of the earth, denoting the user's position with coordinates (x_u, y_u, z_u), and let $\underline{p}_1$, $\underline{p}_2$, and $\underline{p}_3$ represent the three satellites with coordinates (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3), respectively. Then the distance between the user and each of the satellites is

$$|\underline{p}_u - \underline{p}_i| = \sqrt{(x_u - x_i)^2 + (y_u - y_i)^2 + (z_u - z_i)^2} = s_i \quad , \quad i = 1, 2, 3 \quad . \tag{8}$$

Since the time required for the user to receive the timing signals is a function of the distance between the user and the satellite, we can describe the hyperboloid on which the user is located as the locus of points such that the difference in the distance between that point and two other points is constant. One hyperboloid is determined by satellite 1, satellite 2, and $|s_1 - s_2|$; and another by satellite 2, satellite 3, and $|s_2 - s_3|$. From Eq. (8), the two hyperboloids are

$$f_1(x, y, z) = |s_1 - s_2| = \sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2} - \sqrt{(x - x_2)^2 + (y - y_2)^2 + (z - z_2)^2} \tag{9}$$

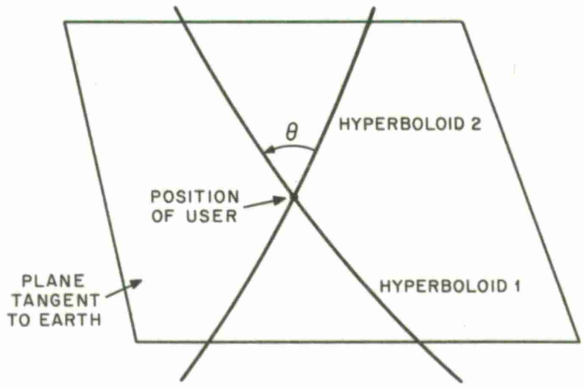
$$f_2(x, y, z) = |s_2 - s_3| = \sqrt{(x - x_2)^2 + (y - y_2)^2 + (z - z_2)^2} - \sqrt{(x - x_3)^2 + (y - y_3)^2 + (z - z_3)^2} \quad . \tag{10}$$

If f is a function of three independent variables (x, y, z), the gradient of the function is defined as

$$\nabla f = \frac{\partial f}{\partial x}\,\underline{i} + \frac{\partial f}{\partial y}\,\underline{j} + \frac{\partial f}{\partial z}\,\underline{k}$$

where $\underline{i}$, $\underline{j}$, and $\underline{k}$ are unit vectors along the positive x, y, and z axes, respectively. Geometrically, ∇f, evaluated at the point (x_o, y_o, z_o), is a vector whose direction is normal to the level surface [i.e., the set of all points such that $f(x, y, z) = c$] at the point (x_o, y_o, z_o). In particular, $f_1(x, y, z)$ in Eq. (9) defines a family of hyperboloids, the value of $|s_1 - s_2|$ defining the particular hyperboloid. The gradient of f_1 evaluated at a point (x_o, y_o, z_o) will be a vector whose direction is normal to the particular hyperboloid passing through (x_o, y_o, z_o). Likewise, we could find a vector whose direction is normal to the hyperboloid defined by $f_2(x, y, z)$ in Eq. (10) passing through the point (x_o, y_o, z_o).

If ∇f_1 and ∇f_2 are the gradients of the functions f_1 and f_2, then the angle Θ, formed by the intersections of the hyperboloids and the tangent plane of the sphere, can be found by computing the angle projected in the tangent plane between ∇f_1 and ∇f_2 evaluated at the user's position.



Let $\underline{E}$ be a unit vector normal to the plane tangent to the sphere at the point of the user. Then

$$\underline{u}_1 = \frac{\nabla f_1 \times \underline{E}}{|\nabla f_1 \times \underline{E}|} \tag{11}$$

and

$$\underline{u}_2 = \frac{\nabla f_2 \times \underline{E}}{|\nabla f_2 \times \underline{E}|} \tag{12}$$

are unit vectors in the plane tangent to the sphere, and

$$\cos \Theta = \underline{u}_1 \cdot \underline{u}_2$$

or

$$\Theta = \cos^{-1} (\underline{u} \cdot \underline{u}_2) \tag{13}$$

When Eq. (13) is evaluated for satellite-user geometries where navigation accuracy is known to be poor, the values of Θ should be near or exactly zero. In particular, if this expression is evaluated for the satellite geometry used in Sec. III at the user positions that had the maximum RMS position error, that is, when the user is located at any point along the equator, $\cos \Theta$ does equal one or the angle between the hyperbolas in the plane tangent to the earth equals zero (see Fig. 5).

Since it is more difficult to solve this equation for non-equatorial cases, a computer program was written. Input to the program consists of the longitude and latitude of the three

satellites and the user; $1/\sin\Theta$ is the output quantity rather than Θ, since the error magnification is seen in Eq. (7) to depend on $1/\sin\Theta$.

There is a direct correspondence between the user locations resulting in large values of $1/\sin\Theta$ and user locations with large RMS position errors for the same satellite geometry. The expression $1/\sin\Theta$ had a minimum value at 60° latitude for all longitudes and increased steadily until, along the equator, Θ became zero and $1/\sin\Theta$ could not be computed. It was also at 60° latitude that the minimum RMS position error occurred, and along the equator that the maximum occurred.

V. RESULTS

It is interesting to explore satellite-user geometries with less symmetry than the equatorial case of Fig. 5. The two computer programs were run, therefore, for other satellite and user positions.

The following satellite positions provide a case with only east-west symmetry.

	Latitude	Longitude
Satellite 1	0.0	30.0
Satellite 2	10.0	0.0
Satellite 3	0.0	−30.0

Navigation accuracy for user positions from +60° to −60° latitude and from −40° to +40° longitude was studied. A singular region again appeared. The maximum RMS error of the user at −40° longitude occurred at latitude +41°; at 0° longitude, it appeared at 49° latitude; and at 40° longitude, the maximum was at 41° latitude. The effect of moving the satellite north was to move the singular region in that direction while the southern hemisphere became free from any singular points. The actual values of the errors are meaningful only in a relative sense because the region was not sampled finely enough to conclude that the worst point was found. The results of the program to calculate the angle between the hyperboloids for this same geometry again showed the maximum value of $1/\sin\Theta$ occurring at exactly the same user positions as the maximum RMS position errors [see Fig. 6(a-b)].

If satellite 2 is now moved to 15° latitude, 15° longitude, all symmetry is destroyed, and user positions from −40° to +40° longitude and +75° to −75° latitude must be studied. (The increase in the range of latitude is needed because of the more northerly position of the satellite.) Again, there is a narrow locus of singular points with the worst point this time occurring (user longitude 0°) at user latitude 68°. This, however, is nearly out of the region of mutual visibility* of all three satellites [see Fig. 7(a-b)]. The region of singular points could probably be moved entirely out of the region where the user could see all three satellites by further adjustment of the northern satellite.

We can conclude that the region of singular points exists whether the satellites are arranged symmetrically or unsymmetrically. The arrangement of the satellites does, however, have an effect on the location of the region of large errors.

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* In this report, visibility curves are computed using an elevation angle of 7°.

Taking the positions of the satellites to be

	Latitude	Longitude
Satellite 1	0.0	45.0
Satellite 2	0.0	0.0
Satellite 3	0.0	−45.0

we find that the singular points again occur on the equator as expected, and that the maximum values of the RMS position errors are the same along the equator as for the previous equatorial case because of the a priori assumptions. If, however, we compare the RMS values of the user positions off the equator for the 30° and 45° cases, we find that the corresponding values for the 30° case are 2.1 times larger than those for the 45° case, and that $1/\sin\Theta$ is 0.5 times larger for the 30° case [see Fig. 8(a-b)]. The effect of having the equatorial satellites 90° instead of 60° apart was a decrease in the width of the region of large errors. However, the region of mutual visibility when the two satellites are 90° apart is so limited that three satellites would not be sufficient to cover the North Atlantic, for example; whereas, this could be done if the satellites were only 60° apart.

Thus, if high-altitude satellites are to be used for navigation, consideration must be given to arranging the satellites so that the user will not be in a singular region, that is, so that the user will be able to determine his position accurately. But consideration must also be given to making the area of mutual visibility of the user and the satellites large enough for the system to be practical.

VI. SUMMARY

From the preceding, the following conclusions can be drawn:

(1) It appears that all satellite geometries result in long, narrow corridors in which navigation accuracy is poor.

(2) The cause of this singular region is geometric in nature; that is, the two hyperboloids and the sphere which locate the user intersect in such a way as to greatly magnify any error in the user's measurements.

(3) A system with all satellites on the equator always yields a singular region along the equator. Thus a practical system would utilize at least one non-equatorial satellite.

(4) If two of the satellites are on the equator and the third is located far enough north, the singular region can be shifted outside the range of mutual visibility.

(5) If two of the satellites are on the equator, the singular region is located in the same hemisphere as the third satellite; i.e., the opposite hemisphere will be free from singular points.

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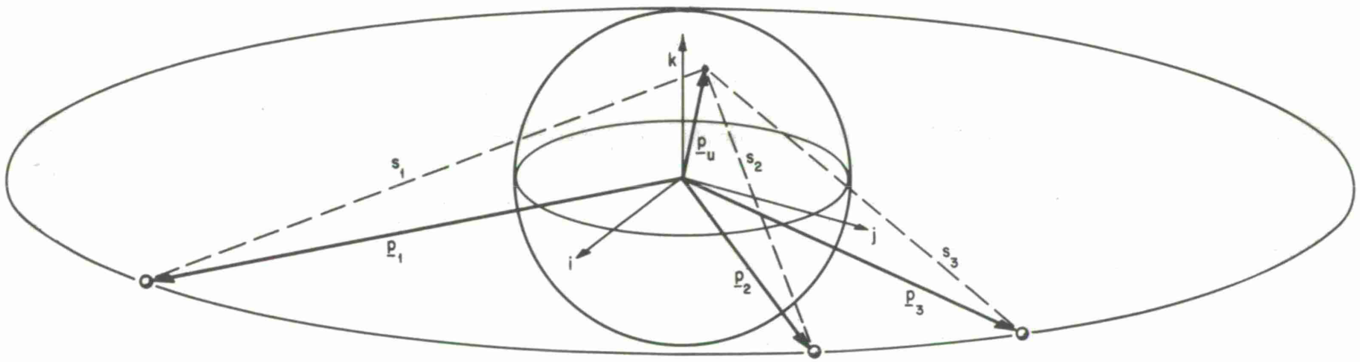


Fig. 1. Hyperbolic navigation system.

USER PCSITICN
LAT LCNG

	0.0	4.C0C	8.D00	12.0DD	16.C00	20.000	24.00D	28.000	32.D00	36.000	40.000
60.0	9.3210 02	S.3190 02	9.3120 D2	9.3020 02	9.2880 02	9.271D 02	9.251D 02	9.2300 02	9.207D D2	9.1850 D2	9.163D 02
59.0	S.3690 02	9.366D 02	S.3590 D2	9.3480 02	9.3320 02	9.3130 02	9.2900 02	9.2660 02	9.239D 02	9.2130 02	9.1870 02
58.C	9.421D 02	9.4180 02	9.410D 02	9.3970 02	9.3800 02	9.3580 02	9.333D 02	9.3050 02	9.2750 02	9.2450 02	9.215D 02
57.0	9.477D 02	9.4740 02	9.466D D2	9.4510 02	9.432D D2	9.407D 02	9.379D 02	9.3480 02	9.3140 02	9.2800 02	9.246D 02
56.0	9.539D 02	9.535D 02	9.526D D2	9.510D 02	9.488D 02	9.461D 02	9.430D 02	9.3950 02	9.357D 02	9.3180 02	9.2790 02
55.0	9.6C50 02	9.601D 02	9.5910 02	9.573D 02	9.5490 C2	9.520D 02	9.4850 02	9.4460 02	9.4040 02	9.3600 02	9.3160 02
54.C	9.676D 02	9.672D 02	9.6600 02	9.6410 02	9.6150 D2	9.583D 02	S.544D 02	9.5010 D2	9.4550 02	9.4060 02	9.3570 02
53.0	9.7520 02	S.7480 02	S.7360 D2	9.7150 02	9.6860 02	9.651D 02	9.6C9D 02	9.5610 02	9.5100 02	9.4560 02	9.4D1D 02
52.0	S.8350 D2	9.830D 02	9.816D 02	9.7940 02	9.763D 02	9.7240 02	9.678D 02	9.6260 02	9.5700 02	9.510D 02	9.4490 02
51.0	9.9230 02	9.918D 02	9.9030 02	9.8780 02	9.845D 02	9.8020 02	9.753D 02	9.6960 02	9.6340 02	9.5690 02	9.502D 02
50.0	1.0020 03	1.0010 03	9.996D 02	9.969D 02	9.933D 02	9.887D 02	9.833D D2	9.7710 02	9.7D4D 02	9.6320 02	9.5580 02
49.0	1.0120 03	1.0110 03	1.CC9D 03	1.007D 03	1.003D 03	9.977D 02	9.919D 02	9.8520 02	9.7790 02	9.7000 02	9.6190 02
48.0	1.0230 03	1.022D 03	1.020D 03	1.017C D3	1.D13D 03	1.007D 03	1.001D 03	9.939D 02	9.859D C2	9.774D 02	9.6850 02
47.C	1.0340 03	1.033D 03	1.0310 D3	1.0280 03	1.024D 03	1.0180 03	1.011D 03	1.0030 D3	9.9450 02	9.853D 02	9.757D 02
46.0	1.046D D3	1.046D 03	1.0430 03	1.0400 03	1.035D 03	1.0290 03	1.022D 03	1.013D D3	1.0040 03	9.938D 02	9.834D 02
45.0	1.D590 03	1.D590 03	1.0560 03	1.053D 03	1.0470 03	1.041D 03	1.0330 03	1.0240 03	1.0140 03	1.0030 03	9.916D 02
44.0	1.D730 03	1.073D 03	1.070D 03	1.066D 03	1.0600 D3	1.0530 03	1.0450 03	1.0350 03	1.0240 03	1.D130 03	1.0010 03
43.0	1.D88D 03	1.087D 03	1.085D 03	1.080D 03	1.0750 03	1.D670 03	1.058D 03	1.0480 03	1.036D 03	1.023D 03	1.D100 03
42.0	1.104D 03	1.103D 03	1.1D0D D3	1.C96D 03	1.0900 03	1.0810 03	1.0720 03	1.0610 03	1.0480 03	1.0350 D3	1.020D 03
41.0	1.121D 03	1.120D 03	1.1170 03	1.1120 03	1.1060 C3	1.097D 03	1.C87D 03	1.0750 03	1.0610 03	1.047D 03	1.031D 03
40.0	1.1390 03	1.1380 03	1.135D D3	1.13D0 03	1.1230 03	1.113D D3	1.102D 03	1.090D 03	1.075D 03	1.060D 03	1.D43D 03
39.0	1.1590 C3	1.157D 03	1.154D 03	1.149D 03	1.1410 03	1.1310 D3	1.11S0 03	1.106D 03	1.091D 03	1.074D 03	1.056D 03
38.0	1.1790 D3	1.178D 03	1.174D 03	1.169D D3	1.1600 03	1.1500 03	1.138D D3	1.123D 03	1.1070 03	1.C89D D3	1.D700 03
37.C	1.2010 03	1.200D 03	1.1S6D 03	1.1900 03	1.1810 03	1.1700 03	1.157D 03	1.142D 03	1.124D 03	1.105D 03	1.085D 03
36.0	1.2250 C3	1.2230 03	1.219D 03	1.2130 D3	1.2D40 03	1.1920 03	1.178D 03	1.162D 03	1.143D 03	1.1230 03	1.1010 03
35.0	1.2500 03	1.2490 03	1.244D D3	1.237C 03	1.2280 03	1.215D D3	1.2C0D 03	1.1830 03	1.1630 03	1.1410 03	1.1180 03
34.C	1.277D 03	1.2760 03	1.2710 D3	1.264D 03	1.253D 03	1.24D0 03	1.224D D3	1.2D6D 03	1.185Q 03	1.1620 03	1.1370 03
33.D	1.3060 03	1.304D 03	1.3CDD 03	1.2920 03	1.2810 03	1.2670 03	1.250D 03	1.231D 03	1.2080 03	1.1830 03	1.1570 03
32.0	1.3370 03	1.3360 03	1.331D 03	1.322D 03	1.311D 03	1.2960 03	1.278D 03	1.257D 03	1.233D 03	1.207D 03	1.1780 03
31.0	1.371D 03	1.369D 03	1.364D 03	1.355D 03	1.3430 03	1.3270 03	1.308D 03	1.286D 03	1.2610 03	1.2320 D3	1.202D 03
3D.0	1.407D 03	1.4050 03	1.399D 03	1.390D 03	1.3770 03	1.3610 03	1.340D C3	1.317D 03	1.2900 03	1.2600 03	1.2270 03
29.0	1.446D 03	1.4440 03	1.438D D3	1.428D 03	1.4140 03	1.3970 03	1.375D 03	1.3500 03	1.3220 03	1.2900 03	1.255D 03
28.0	1.4880 03	1.4860 03	1.4790 03	1.4690 03	1.455D C3	1.4360 03	1.4130 03	1.3870 03	1.356D 03	1.322D 03	1.2850 03
27.D	1.533D 03	1.5310 03	1.524D 03	1.513D D3	1.498D 03	1.478D 03	1.4540 03	1.426D D3	1.394D 03	1.3570 03	1.318D 03
26.0	1.5830 03	1.5800 03	1.573D 03	1.5620 03	1.545D 03	1.525D 03	1.4990 D3	1.469D 03	1.434D 03	1.3960 03	1.3530 03
25.0	1.636D D3	1.6340 03	1.627D 03	1.614D 03	1.597D 03	1.575D 03	1.548D 03	1.516D D3	1.479D 03	1.438D 03	1.3930 03
24.0	1.6950 03	1.6S3D 03	1.685D 03	1.672D 03	1.653D 03	1.6300 03	1.601D 03	1.5670 D3	1.5280 03	1.484D 03	1.4350 03
23.0	1.759D 03	1.757D 03	1.748D 03	1.734D 03	1.715D 03	1.69DD D3	1.659D 03	1.623D 03	1.5810 03	1.534D 03	1.483D 03
22.0	1.830D 03	1.827D 03	1.818D 03	1.8030 03	1.7830 03	1.7560 03	1.723D 03	1.685D 03	1.640D 03	1.5900 03	1.5350 03
21.0	1.9D7D 03	1.904D 03	1.895D D3	1.879D 03	1.8570 C3	1.8290 03	1.7940 03	1.753D 03	1.7050 03	1.6520 03	1.5920 03
20.0	1.993D 03	1.9900 03	1.9800 03	1.963D D3	1.9400 03	1.910D 03	1.873D 03	1.829D 03	1.778D 03	1.7200 03	1.657D 03
1S.0	2.0890 03	2.D850 03	2.075D 03	2.D57D 03	2.032D 03	1.999D 03	1.9600 03	1.913D 03	1.858D 03	1.7970 03	1.7280 03
18.0	2.1950 03	2.1S2D 03	2.180D 03	2.1610 03	2.134D 03	2.1000 03	2.0570 03	2.007D 03	1.948D D3	1.8820 03	1.809D 03
17.0	2.315D D3	2.311D 03	2.2990 03	2.278D 03	2.25D0 03	2.2130 03	2.167D 03	2.113D 08	2.05DD 03	1.9790 03	1.8990 03
16.0	2.450D D3	2.4460 03	2.4330 03	2.411D 03	2.3800 03	2.3400 D3	2.2910 03	2.233D D3	2.1650 03	2.088D 03	2.002D 03
15.0	2.6040 03	2.600D 03	2.585D 03	2.562D D3	2.5280 03	2.485D 03	2.4320 03	2.369D 03	2.296D 03	2.213D 03	2.1200 03
14.0	2.781D C3	2.776D 03	2.760D 03	2.735D D3	2.699D C3	2.652D 03	2.595D 03	2.526D 03	2.4470 03	2.356D 03	2.2550 03
13.0	2.985D 03	2.980D 03	2.9630 03	2.9350 03	2.8960 03	2.8450 03	2.7830 03	2.708D 03	2.6220 03	2.523D 03	2.4120 03
12.0	3.225D 03	3.2190 03	3.2C0D 03	3.170D 03	3.127D 03	3.072D 03	3.0C30 03	2.922D 03	2.827D D3	2.7180 03	2.5970 03
11.0	3.5080 03	3.5020 03	3.482D 03	3.44ED 03	3.401D 03	3.340D D3	3.265D 03	3.175D 03	3.07D0 03	2.9510 03	2.8170 03
10.0	3.8490 03	3.842D 03	3.8200 03	3.7830 03	3.7310 03	3.6630 03	3.580D 03	3.48DD C3	3.3640 D3	3.231D D3	3.082D 03
9.0	4.2680 03	4.259D 03	4.2350 03	4.1930 03	4.1350 03	4.059D 03	3.9660 03	3.854D 03	3.724D 03	3.575D 03	3.4080 03
8.0	4.7910 03	4.7820 D3	4.754D C3	4.7070 03	4.641D 03	4.5550 03	4.45DD 03	4.3230 03	4.176D D3	4.0070 03	3.817D D3
7.0	5.4660 C3	5.455D 03	5.423D 03	5.3690 D3	5.2940 03	5.1950 03	5.073D 03	4.928D 03	4.758D 03	4.564D 03	4.3450 03
6.0	6.3670 03	6.355D 03	6.3170 03	6.2540 03	6.1650 03	6.0500 03	5.907D 03	5.736D D3	5.537D 03	5.3090 03	5.052D 03
5.0	7.6310 03	7.616D 03	7.5700 03	7.4940 03	7.387D 03	7.248D 03	7.0760 03	6.871D 03	6.6310 03	6.356D 03	6.0450 03
4.0	9.5280 03	9.5090 03	9.4520 03	9.3570 03	9.2230 03	9.0480 03	8.8330 03	8.575D 03	8.2740 03	7.S280 03	7.5380 03
3.0	1.2690 04	1.267D 04	1.259D 04	1.246D 04	1.2280 04	1.2050 04	1.1760 04	1.1420 D4	1.102D 04	1.055D 04	1.0030 04
2.0	1.903D 04	1.899D 04	1.887D C4	1.8680 04	1.841D 04	1.806D 04	1.763D 04	1.711D D4	1.651D 04	1.581D 04	1.5030 04
1.0	3.8020 04	3.7940 04	3.7710 04	3.733D 04	3.6790 04	3.6090 04	3.5230 04	3.4190 D4	3.298D 04	3.159D 04	3.0020 04
0.0	1.000D 06	1.000D 06	1.000D 06	1.000D 06	1.000D 06	1.0000 06	1.0D0D 06	1.000D 06	1.000D 06	1.000D 06	1.0000 06

Fig. 2. RMS position errors for all equatorial satellites. (All values in meters.)

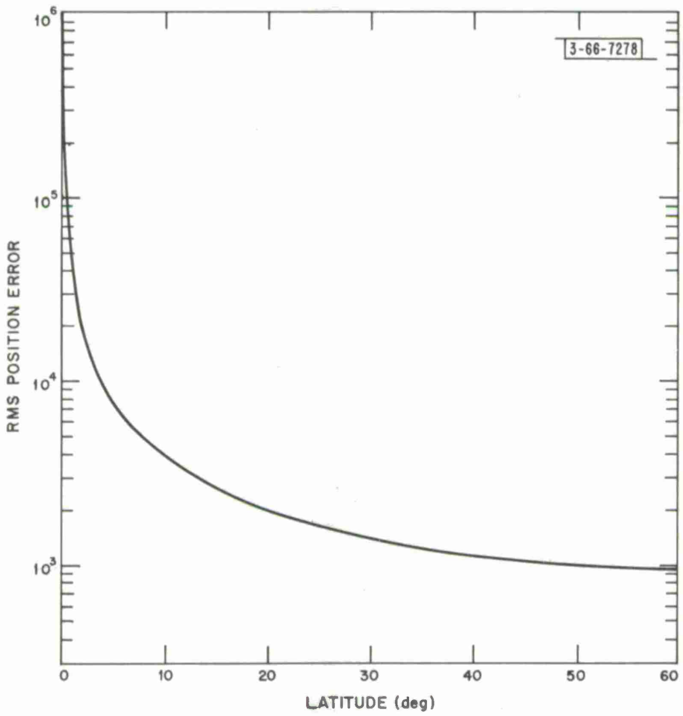


Fig. 3. Cross section of RMS position error vs latitude (longitude = 0°).

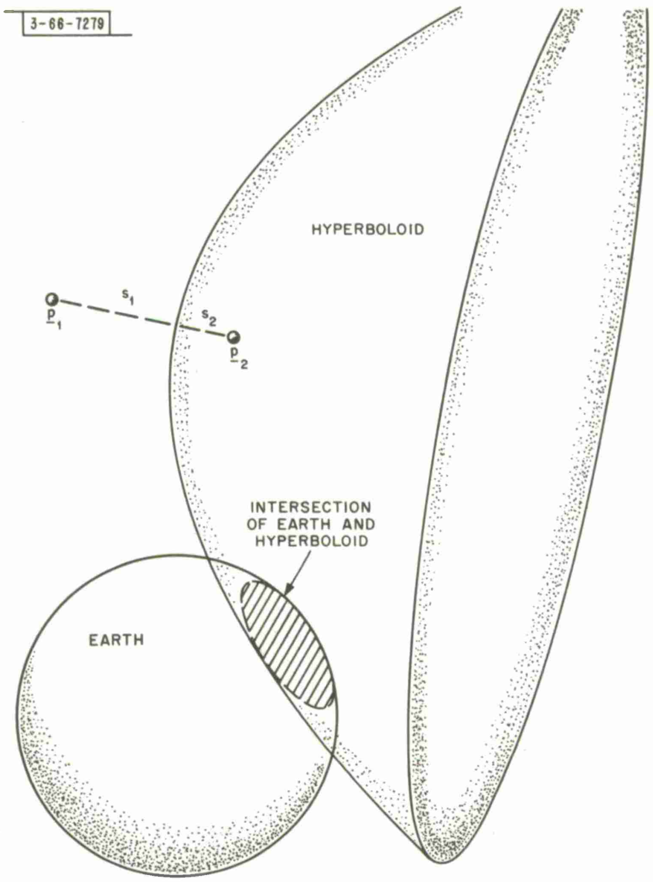


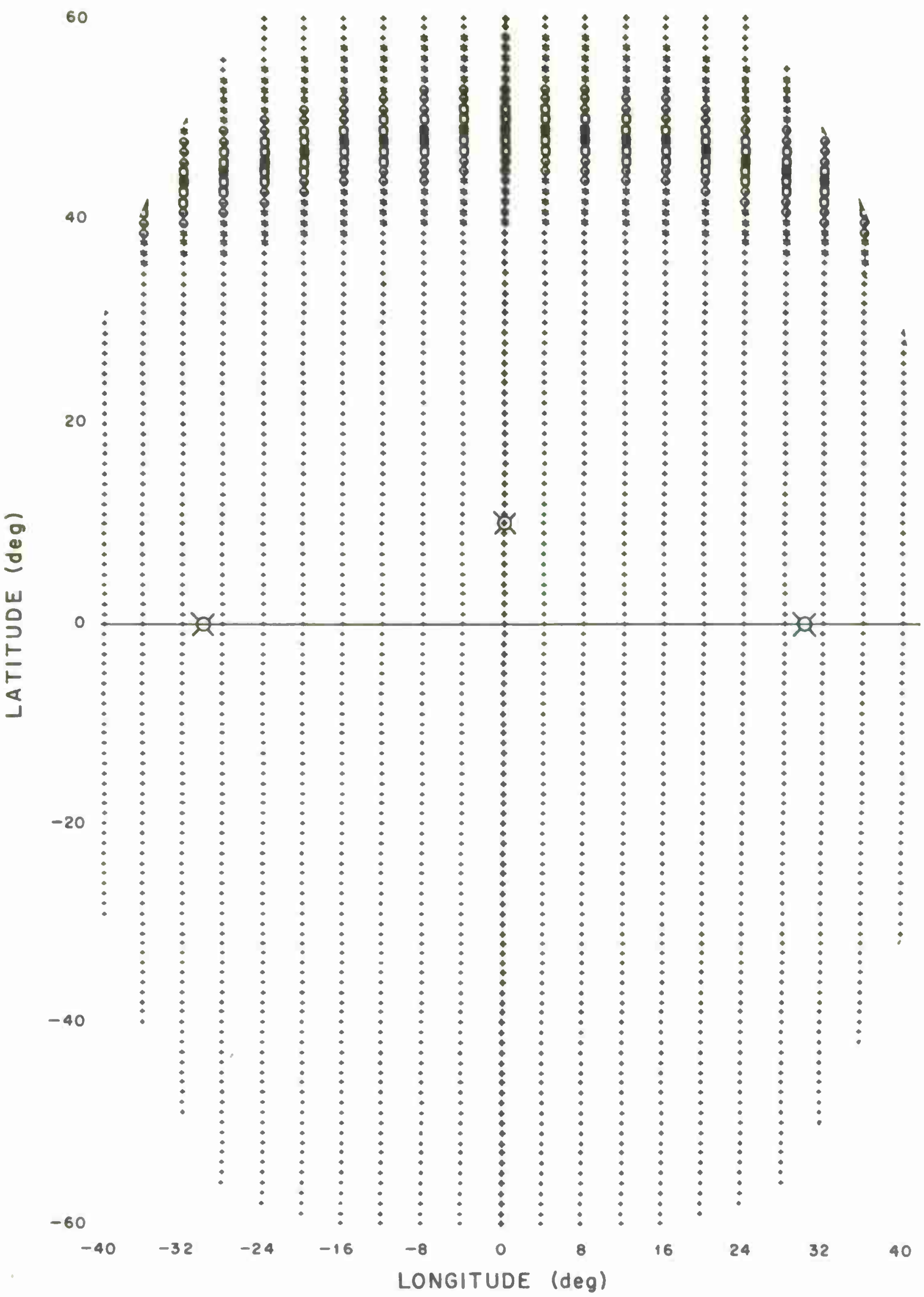
Fig. 4. Intersection of hyperboloid with tangent plane of earth.

-66-7280

USER POSITION

LAT \ LONG	0.0	4.00	8.00	12.00	16.00	20.00	24.00	28.00	32.00	36.00	40.00
60.00	2.110E 00	2.107E 00	2.101E 00	2.091E 00	2.076E 00	2.059E 00	2.038E 00	2.015E 00	1.990E 00	1.964E 00	1.937E 00
59.00	2.124E 00	2.121E 00	2.114E 00	2.103E 00	2.088E 00	2.068E 00	2.046E 00	2.021E 00	1.994E 00	1.966E 00	1.937E 00
58.00	2.138E 00	2.136E 00	2.128E 00	2.116E 00	2.100E 00	2.079E 00	2.055E 00	2.028E 00	1.999E 00	1.968E 00	1.937E 00
57.00	2.154E 00	2.152E 00	2.144E 00	2.130E 00	2.112E 00	2.090E 00	2.064E 00	2.035E 00	2.004E 00	1.971E 00	1.937E 00
56.00	2.171E 00	2.168E 00	2.160E 00	2.145E 00	2.126E 00	2.102E 00	2.074E 00	2.043E 00	2.009E 00	1.974E 00	1.938E 00
55.00	2.189E 00	2.186E 00	2.177E 00	2.161E 00	2.141E 00	2.115E 00	2.085E 00	2.051E 00	2.015E 00	1.977E 00	1.939E 00
54.00	2.208E 00	2.205E 00	2.195E 00	2.178E 00	2.156E 00	2.129E 00	2.096E 00	2.060E 00	2.022E 00	1.981E 00	1.940E 00
53.00	2.228E 00	2.225E 00	2.214E 00	2.196E 00	2.173E 00	2.143E 00	2.109E 00	2.070E 00	2.029E 00	1.985E 00	1.941E 00
52.00	2.250E 00	2.246E 00	2.234E 00	2.216E 00	2.190E 00	2.159E 00	2.122E 00	2.081E 00	2.036E 00	1.990E 00	1.943E 00
51.00	2.272E 00	2.268E 00	2.256E 00	2.236E 00	2.209E 00	2.175E 00	2.136E 00	2.092E 00	2.045E 00	1.995E 00	1.945E 00
50.00	2.296E 00	2.292E 00	2.279E 00	2.258E 00	2.229E 00	2.193E 00	2.151E 00	2.104E 00	2.053E 00	2.001E 00	1.947E 00
49.00	2.322E 00	2.317E 00	2.303E 00	2.281E 00	2.250E 00	2.211E 00	2.167E 00	2.117E 00	2.063E 00	2.007E 00	1.950E 00
48.00	2.349E 00	2.344E 00	2.329E 00	2.305E 00	2.272E 00	2.231E 00	2.184E 00	2.131E 00	2.074E 00	2.014E 00	1.953E 00
47.00	2.378E 00	2.372E 00	2.357E 00	2.331E 00	2.296E 00	2.252E 00	2.202E 00	2.146E 00	2.085E 00	2.022E 00	1.957E 00
46.00	2.408E 00	2.402E 00	2.386E 00	2.358E 00	2.321E 00	2.275E 00	2.221E 00	2.161E 00	2.097E 00	2.030E 00	1.961E 00
45.00	2.440E 00	2.434E 00	2.417E 00	2.388E 00	2.348E 00	2.299E 00	2.242E 00	2.178E 00	2.110E 00	2.039E 00	1.966E 00
44.00	2.475E 00	2.468E 00	2.449E 00	2.419E 00	2.377E 00	2.325E 00	2.264E 00	2.197E 00	2.124E 00	2.049E 00	1.972E 00
43.00	2.511E 00	2.504E 00	2.484E 00	2.451E 00	2.407E 00	2.352E 00	2.288E 00	2.216E 00	2.140E 00	2.059E 00	1.978E 00
42.00	2.549E 00	2.542E 00	2.521E 00	2.486E 00	2.439E 00	2.381E 00	2.313E 00	2.237E 00	2.156E 00	2.071E 00	1.985E 00
41.00	2.590E 00	2.583E 00	2.560E 00	2.524E 00	2.474E 00	2.412E 00	2.340E 00	2.260E 00	2.174E 00	2.084E 00	1.993E 00
40.00	2.634E 00	2.626E 00	2.602E 00	2.563E 00	2.510E 00	2.445E 00	2.369E 00	2.284E 00	2.193E 00	2.098E 00	2.002E 00
39.00	2.680E 00	2.672E 00	2.647E 00	2.605E 00	2.549E 00	2.480E 00	2.400E 00	2.310E 00	2.214E 00	2.113E 00	2.012E 00
38.00	2.729E 00	2.721E 00	2.694E 00	2.650E 00	2.591E 00	2.518E 00	2.433E 00	2.338E 00	2.236E 00	2.130E 00	2.023E 00
37.00	2.782E 00	2.773E 00	2.744E 00	2.698E 00	2.636E 00	2.558E 00	2.468E 00	2.368E 00	2.260E 00	2.148E 00	2.035E 00
36.00	2.838E 00	2.828E 00	2.798E 00	2.750E 00	2.683E 00	2.601E 00	2.506E 00	2.400E 00	2.286E 00	2.168E 00	2.048E 00
35.00	2.898E 00	2.887E 00	2.856E 00	2.804E 00	2.734E 00	2.648E 00	2.547E 00	2.435E 00	2.314E 00	2.189E 00	2.063E 00
34.00	2.962E 00	2.951E 00	2.917E 00	2.863E 00	2.789E 00	2.697E 00	2.591E 00	2.472E 00	2.345E 00	2.213E 00	2.079E 00
33.00	3.030E 00	3.019E 00	2.983E 00	2.926E 00	2.848E 00	2.751E 00	2.638E 00	2.513E 00	2.378E 00	2.238E 00	2.097E 00
32.00	3.104E 00	3.091E 00	3.054E 00	2.993E 00	2.911E 00	2.808E 00	2.689E 00	2.557E 00	2.414E 00	2.266E 00	2.117E 00
31.00	3.183E 00	3.170E 00	3.130E 00	3.066E 00	2.978E 00	2.870E 00	2.744E 00	2.604E 00	2.453E 00	2.297E 00	2.139E 00
30.00	3.268E 00	3.254E 00	3.212E 00	3.144E 00	3.051E 00	2.937E 00	2.804E 00	2.655E 00	2.496E 00	2.331E 00	2.164E 00
29.00	3.359E 00	3.344E 00	3.300E 00	3.228E 00	3.130E 00	3.009E 00	2.868E 00	2.711E 00	2.543E 00	2.368E 00	2.191E 00
28.00	3.458E 00	3.442E 00	3.395E 00	3.319E 00	3.216E 00	3.087E 00	2.938E 00	2.772E 00	2.594E 00	2.408E 00	2.221E 00
27.00	3.564E 00	3.548E 00	3.498E 00	3.418E 00	3.308E 00	3.172E 00	3.014E 00	2.838E 00	2.649E 00	2.453E 00	2.255E 00
26.00	3.680E 00	3.662E 00	3.610E 00	3.525E 00	3.409E 00	3.265E 00	3.097E 00	2.911E 00	2.710E 00	2.502E 00	2.292E 00
25.00	3.806E 00	3.787E 00	3.732E 00	3.641E 00	3.518E 00	3.365E 00	3.188E 00	2.990E 00	2.777E 00	2.557E 00	2.334E 00
24.00	3.943E 00	3.923E 00	3.865E 00	3.768E 00	3.638E 00	3.476E 00	3.287E 00	3.077E 00	2.852E 00	2.617E 00	2.380E 00
23.00	4.093E 00	4.072E 00	4.010E 00	3.908E 00	3.769E 00	3.597E 00	3.396E 00	3.173E 00	2.933E 00	2.684E 00	2.432E 00
22.00	4.258E 00	4.235E 00	4.169E 00	4.060E 00	3.913E 00	3.730E 00	3.517E 00	3.279E 00	3.024E 00	2.759E 00	2.490E 00
21.00	4.439E 00	4.415E 00	4.345E 00	4.229E 00	4.072E 00	3.877E 00	3.650E 00	3.397E 00	3.125E 00	2.842E 00	2.556E 00
20.00	4.639E 00	4.614E 00	4.539E 00	4.416E 00	4.248E 00	4.040E 00	3.798E 00	3.528E 00	3.238E 00	2.935E 00	2.630E 00
19.00	4.862E 00	4.835E 00	4.754E 00	4.623E 00	4.444E 00	4.222E 00	3.963E 00	3.674E 00	3.364E 00	3.041E 00	2.714E 00
18.00	5.110E 00	5.081E 00	4.995E 00	4.855E 00	4.663E 00	4.425E 00	4.148E 00	3.839E 00	3.507E 00	3.160E 00	2.809E 00
17.00	5.389E 00	5.358E 00	5.266E 00	5.115E 00	4.909E 00	4.654E 00	4.357E 00	4.025E 00	3.668E 00	3.296E 00	2.918E 00
16.00	5.704E 00	5.671E 00	5.572E 00	5.409E 00	5.188E 00	4.914E 00	4.594E 00	4.237E 00	3.852E 00	3.451E 00	3.044E 00
15.00	6.063E 00	6.027E 00	5.920E 00	5.745E 00	5.506E 00	5.210E 00	4.864E 00	4.478E 00	4.063E 00	3.629E 00	3.189E 00
14.00	6.474E 00	6.435E 00	6.319E 00	6.129E 00	5.871E 00	5.550E 00	5.176E 00	4.758E 00	4.307E 00	3.836E 00	3.358E 00
13.00	6.950E 00	6.908E 00	6.782E 00	6.575E 00	6.294E 00	5.945E 00	5.537E 00	5.082E 00	4.591E 00	4.078E 00	3.557E 00
12.00	7.508E 00	7.461E 00	7.323E 00	7.098E 00	6.790E 00	6.408E 00	5.962E 00	5.464E 00	4.927E 00	4.365E 00	3.793E 00
11.00	8.168E 00	8.117E 00	7.965E 00	7.717E 00	7.378E 00	6.958E 00	6.467E 00	5.919E 00	5.327E 00	4.708E 00	4.077E 00
10.00	8.963E 00	8.906E 00	8.738E 00	8.463E 00	8.087E 00	7.621E 00	7.077E 00	6.469E 00	5.812E 00	5.123E 00	4.422E 00
9.00	9.936E 00	9.873E 00	9.685E 00	9.377E 00	8.956E 00	8.435E 00	7.826E 00	7.145E 00	6.409E 00	5.637E 00	4.851E 00
8.00	1.116E 01	1.108E 01	1.087E 01	1.052E 01	1.005E 01	9.456E 00	8.767E 00	7.994E 00	7.161E 00	6.286E 00	5.393E 00
7.00	1.273E 01	1.265E 01	1.240E 01	1.200E 01	1.145E 01	1.077E 01	9.980E 00	9.093E 00	8.134E 00	7.127E 00	6.099E 00
6.00	1.483E 01	1.473E 01	1.444E 01	1.397E 01	1.333E 01	1.254E 01	1.160E 01	1.056E 01	9.439E 00	8.257E 00	7.049E 00
5.00	1.777E 01	1.765E 01	1.731E 01	1.674E 01	1.597E 01	1.501E 01	1.389E 01	1.263E 01	1.127E 01	9.849E 00	8.392E 00
4.00	2.219E 01	2.204E 01	2.161E 01	2.090E 01	1.993E 01	1.872E 01	1.732E 01	1.575E 01	1.404E 01	1.225E 01	1.042E 01
3.00	2.956E 01	2.938E 01	2.879E 01	2.784E 01	2.654E 01	2.493E 01	2.306E 01	2.095E 01	1.866E 01	1.628E 01	1.382E 01
2.00	4.431E 01	4.402E 01	4.316E 01	4.173E 01	3.979E 01	3.737E 01	3.453E 01	3.138E 01	2.794E 01	2.433E 01	2.067E 01
1.00	8.875E 01	8.838E 01	8.628E 01	8.368E 01	7.963E 01	7.488E 01	6.916E 01	6.276E 01	5.589E 01	4.863E 01	4.125E 01
0.0	ZERO ANGLE	ZERO ANGLE	ZERO ANGLE	ZERO ANGLE	ZERO ANGLE	ZERO ANGLE	ZERO ANGLE	ZERO ANGLE	ZERO ANGLE	ZERO ANGLE	ZERO ANGLE

Fig. 5. $1/\sin\theta$ for all equatorial satellites.



VALUES OF Q IN THE RANGE

+	< 60
✱	60 – 69
⊙	70 – 79
□	80 – 100
⊠	> 100 AND $\theta = 0°$

$$\text{WHERE } Q = \frac{100}{3} \text{ LOG} \left(\frac{1}{\sin\theta}\right) + 33$$

-66-7281

Fig. 6(a). $1/\sin\theta$ for an east–west symmetric satellite geometry.

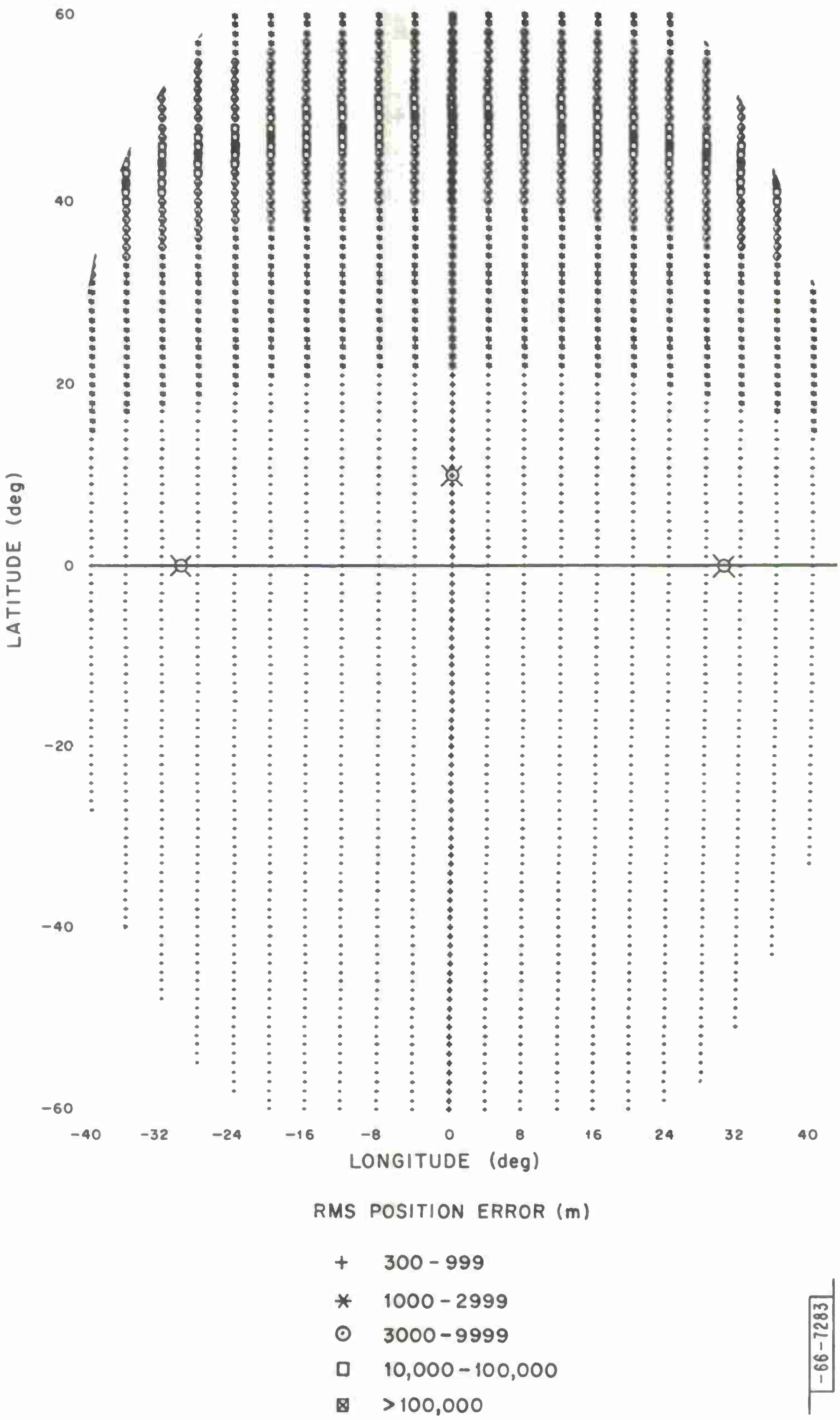
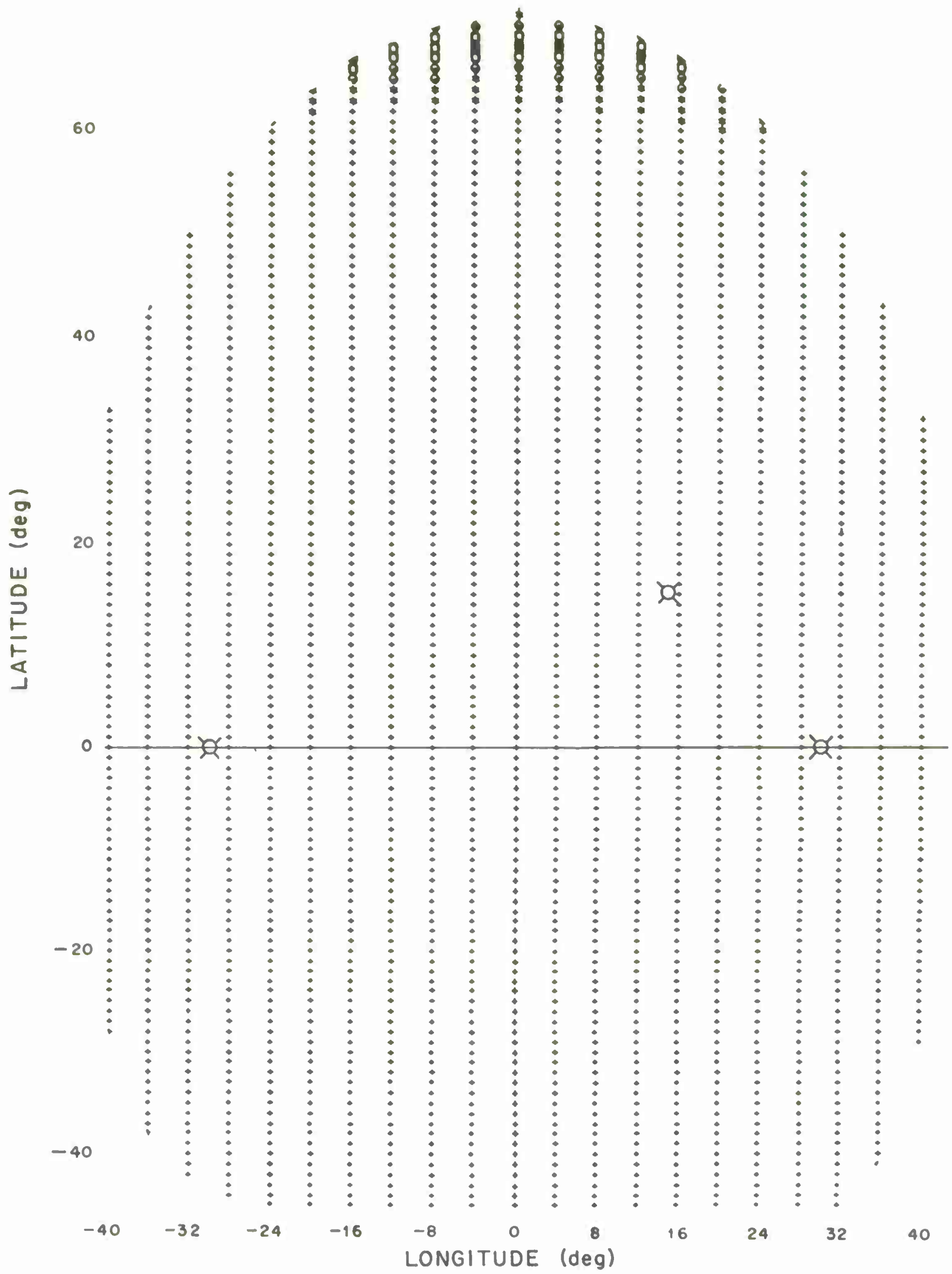


Fig. 6(b). RMS position errors for an east-west symmetric satellite geometry.



VALUES OF Q IN THE RANGE

+	< 60
✱	60 – 69
⊙	70 – 79
□	80 – 100
⊠	> 100 AND $\theta = 0°$

$$\text{WHERE } Q = \frac{100}{3} \text{ LOG} \left(\frac{1}{\sin\theta}\right) + 33$$

-66-7284

Fig. 7(a). $1/\sin\theta$ for an unsymmetric satellite geometry.

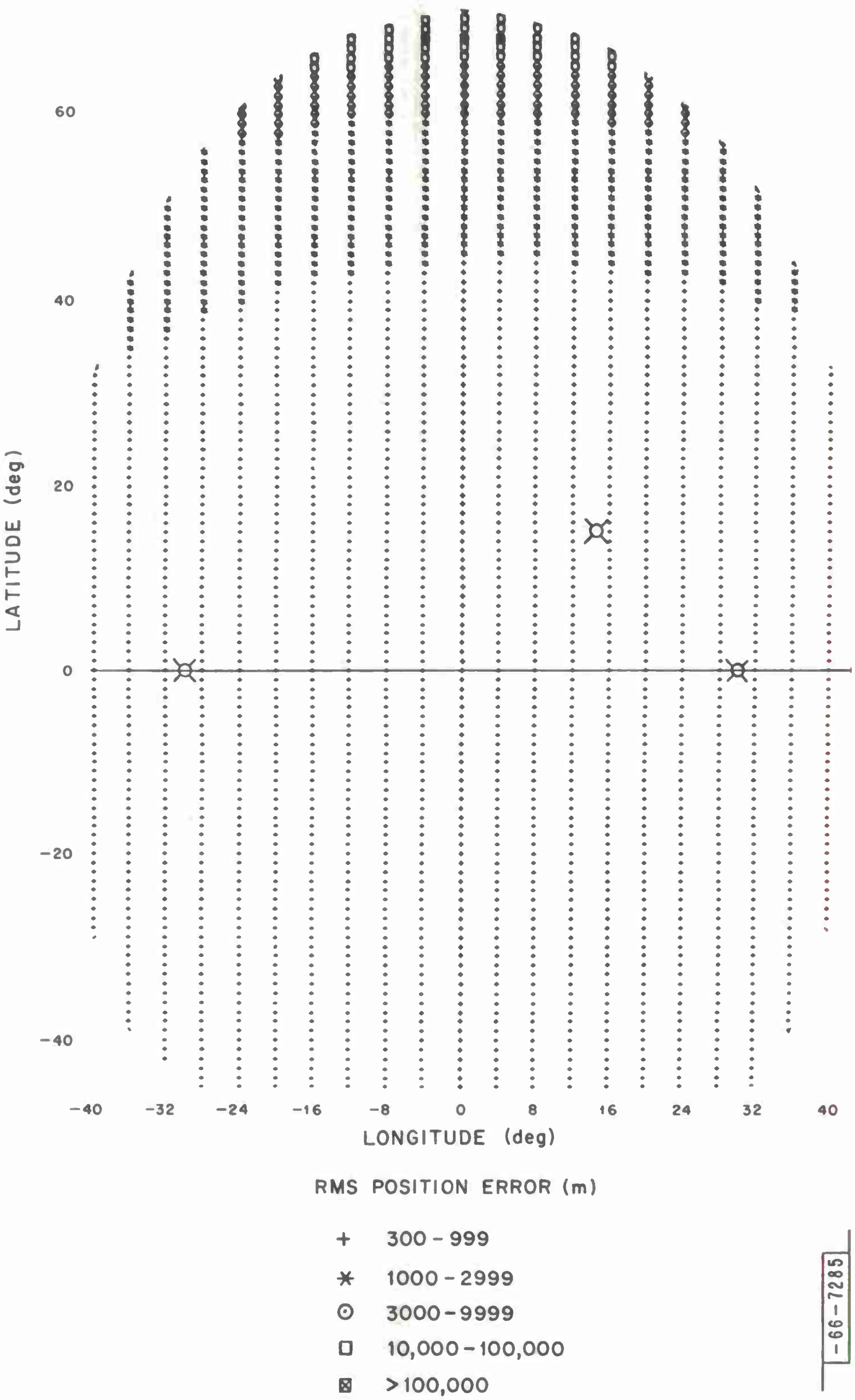


Fig. 7(b). RMS position errors for an unsymmetric satellite geometry.

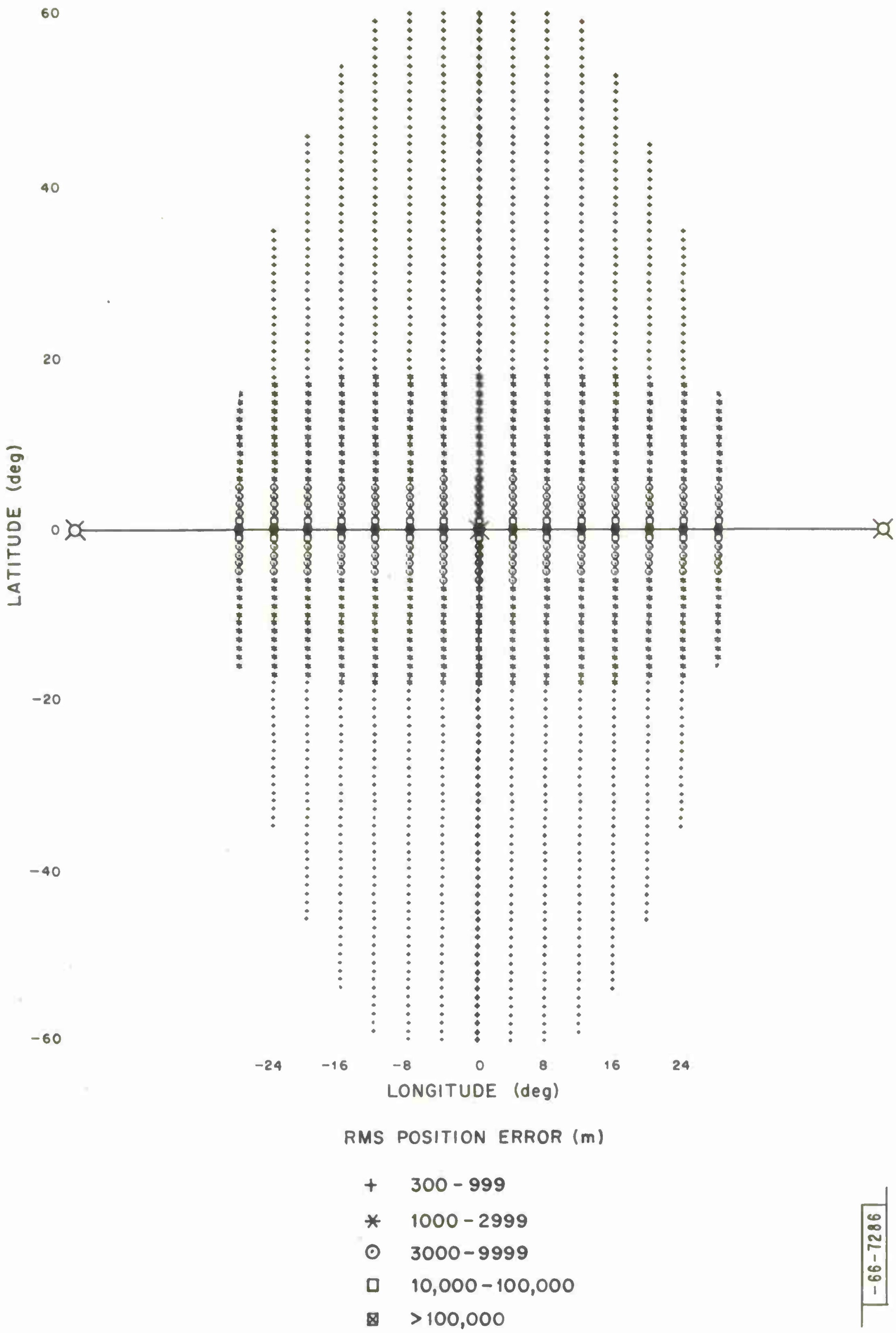


Fig. 8(a). RMS position errors for satellites at 45-0, 0-0, −45-0.

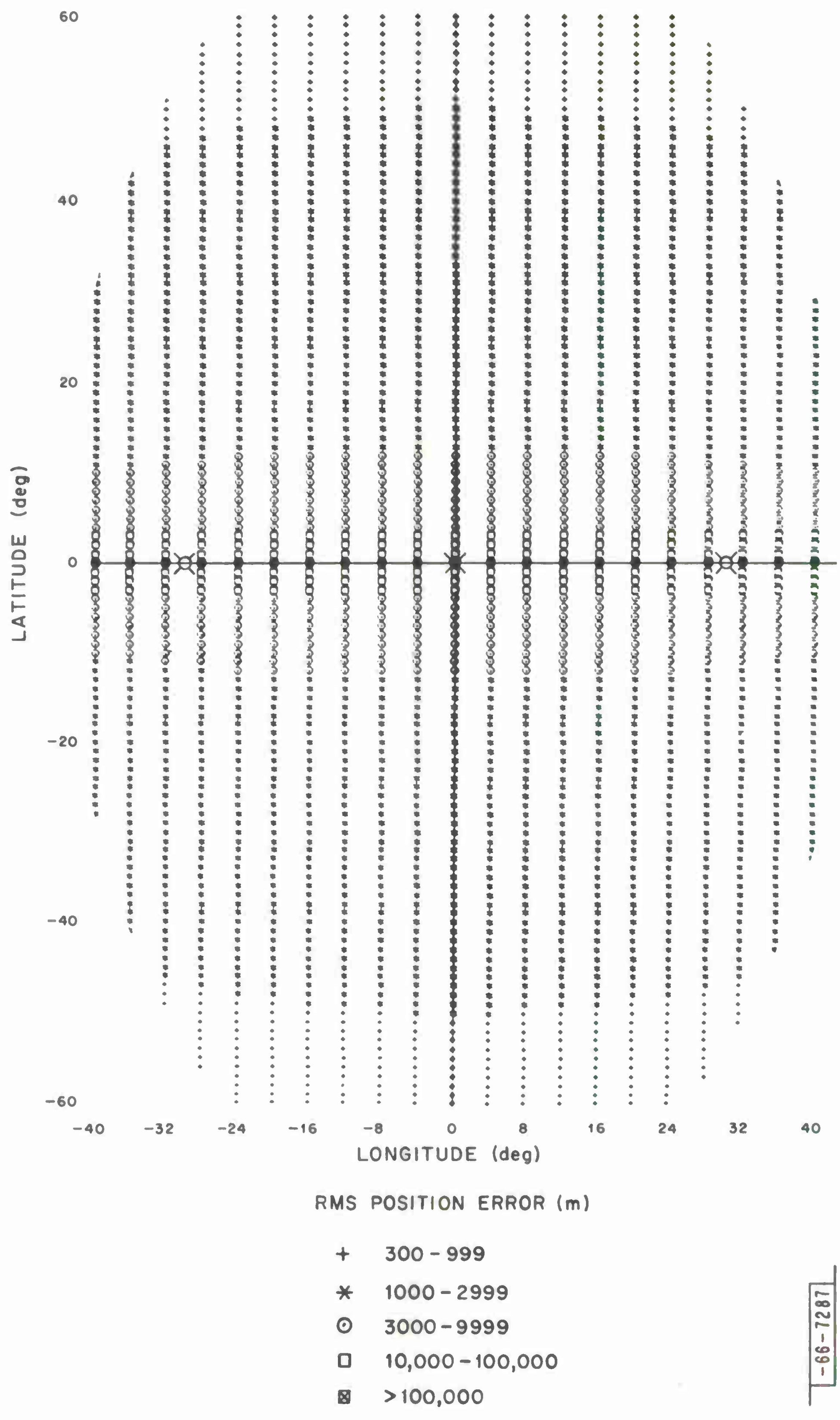


Fig. 8(b). RMS position errors for satellites at 30-0, 0-0, −30-0.

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13. ABSTRACT

An error analysis of an hyperbolic navigation system using high-altitude satellites revealed that certain satellite arrangements result in long, narrow corridors on the earth (singular regions), in which navigation errors are very large. The study showed that in these singular regions the satellite-user geometry results in navigation equations which are sensitive to measurement errors and thus cannot be solved accurately for all three user coordinates.

14. KEY WORDS

hyperbolic navigation system   singular region   satellite-user geometry

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